International Journal of Theoretical Physics, Vol. 26, No. 2, 1987

Electrodynamics from a Metric

R. G. Beil[1]

Received September 24, 1986

A metric is given that produces a space in which the geodesic equation is identical with the Lorentz equation of motion for a charged particle. The gravitational field equations in the same space indicate a geometric origin for the electromagnetic energy-momentum tensor. A comparison is made with Kaluza-Klein theories and it is determined that the present theory is distinct from them because it corresponds to a timelike, noncompact fifth dimension. Since the metric is velocity-dependent, it is actually a Finsler space rather than a Riemannian space metric. Its special form, however, allows computations to be done in terms of Riemannian geometry.

1. THE EQUATION OF MOTION

Consider a charged test particle moving along a path in Minkowski space. The path parameter is taken to be the proper time τ. The position of the particle is given by $x^{\mu}(\tau)$ and the velocity and acceleration are $v^{\mu}=dx^{\mu}/d\tau$ and $a^{\mu}=dv^{\mu}/d\tau$. Take

$$c^2\,d\tau^2=\eta_{\mu\nu}dx^{\mu}\,dx^{\nu}$$

where the signature of the metric is $(+1, -1, -1, -1)$.

When the particle is acted on only by an electromagnetic field from a potential A^{μ}, assume the equation of motion is of the Lorentz form

$$a^{\mu}=(e/mc)\eta^{\mu\nu}F_{\nu\lambda}v^{\lambda} \tag{1}$$

with $F_{\mu\nu}=A_{\nu,\mu}-A_{\mu,\nu}$.

This path is not a geodesic in Minkowski space. It will be shown, however, that the behavior of the particle can also be described by assuming it follows a geodesic in a space with a metric $\bar{g}_{\mu\nu}$ of a certain form. The electromagnetic force then results from the connection produced by the new metric.

[1]313 S. Washington, Marshall, Texas 75670.

0020-7748/87/0200-0189$05.00/0

The metric $\bar{g}_{\mu\nu}$ is imposed on the same coordinate system as the Minkowski metric. The introduction of a new metric is always possible for any differentiable manifold. This does not require a coordinate transformation in either the active or the passive sense. As far as the particle is concerned, the new metric produces a change of scale along its path, which is characterized by a new path parameter $\bar{\tau}$ with

$$c^2\, d\bar{\tau}^2 = \bar{g}_{\mu\nu}\, dx^\mu\, dx^\nu$$

Then, for example,

$$\bar{v}^\mu = dx^\mu / d\bar{\tau} = bv^\mu \tag{2}$$

where $b = d\tau / d\bar{\tau}$ can be thought of as a scale function.

Also,

$$\bar{a}^\mu = d\bar{v}^\mu / d\bar{\tau} = a^\mu b^2 + v^\mu\, db/d\bar{\tau} \tag{3}$$

The key to this work is the Ansatz for the form of $\bar{g}_{\mu\nu}$:

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + kB_\mu B_\nu \tag{4}$$

where k is a constant to be determined. The vector B_μ will be related to the electromagnetic potential.

This type of metric has appeared before in various contexts. For example, it looks like the 4-tensor part of a Kaluza–Klein metric. It is also a type of projection tensor (Schmutzer, 1983). It will be found to be similar to a form that Synge (1971) has discussed as representing a transparent medium. Moreover, a recent effort by Fontaine and Amiot (1983) is in the same general direction as the present one.

The contravariant form must be

$$\bar{g}^{\mu\nu} = \eta^{\mu\nu} - k(1 + kB^2)^{-1} B^\mu B^\nu \tag{5}$$

where $B^2 = B_\alpha B^\alpha$. The vector B^μ is defined in the Minkowski space so its index is raised and lowered by η. The necessary condition that determines the contravariant form is

$$\bar{g}^{\mu\alpha} \bar{g}_{\alpha\nu} = \delta^\mu_\nu \tag{6}$$

In Fontaine and Amiot (1983) the condition (6) is not satisfied.

Now, considering

$$c^2 d\bar{\tau}^2 = \bar{g}_{\mu\nu} v^\mu v^\nu\, d\tau^2 = [c^2 + k(B_\alpha v^\alpha)^2]\, d\tau^2$$

gives

$$b = [1 + kc^{-2}(B_\alpha v^\alpha)^2]^{-1/2} \tag{7}$$

Note that

$$\bar{v}_\mu = b[v_\mu + kB_\mu(B_\alpha v^\alpha)]$$

so that $\bar{v}_\alpha \bar{v}^\alpha = c^2$. Also,

$$db/d\bar{\tau} = b(db/d\tau) = -b^4 kc^{-2}(B_\alpha v^\alpha)\, d(B_\alpha v^\alpha)/d\tau$$

The geodesic equation in the new metric is

$$\bar{a}^\mu + \overline{\left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\}} \bar{v}^\alpha \bar{v}^\beta = 0 \tag{8}$$

A computation of the Christoffel connection gives

$$\overline{\left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\}} = \tfrac{1}{2}k[\bar{g}^{\mu\lambda}(H_{\beta\lambda}B_\alpha + H_{\alpha\lambda}B_\beta) + (1+kB^2)^{-1}(B_{\alpha,\beta} + B_{\beta,\alpha})B^\mu] \tag{9}$$

where $H_{\mu\nu} = B_{\nu,\mu} - B_{\mu,\nu}$.

Using (3), (7), and (9) in (8) and dividing by b^2 results in

$$a^\mu - b^2 kc^{-2} v^\mu (B_\alpha v^\alpha)\, d(B_\alpha v^\alpha)/d\tau + k(1+kB^2)^{-1} B^\mu (v^\alpha\, dB_\alpha/d\tau) + k\bar{g}^{\mu\lambda} H_{\beta\lambda}(B_\alpha v^\alpha) v^\beta = 0 \tag{10}$$

A useful simplification results from inserting

$$v^\alpha\, dB_\alpha/d\tau = d(B_\alpha v^\alpha)/d\tau - B_\alpha a^\alpha$$

in the third term of (10), multiplying the entire equation by B_μ, solving for $B_\alpha a^\alpha$, and substituting the result back in (10).

The geodesic equation is then

$$a^\mu + k[B^\mu - c^{-2}(B_\alpha v^\alpha)v^\mu]b^2\, d(B_\alpha v^\alpha)/d\tau + k\eta^{\mu\lambda} H_{\beta\lambda}(B_\alpha v^\alpha) v^\beta = 0 \tag{11}$$

Now assume that B_μ is related to the electromagnetic potential by a gauge,

$$B_\mu = A_\mu + \partial\Lambda/\partial x^\mu \tag{12}$$

Then $H_{\mu\nu} = F_{\mu\nu}$.

It is easy to see that if the condition

$$k(B_\alpha v^\alpha) = -e/mc \tag{13}$$

is satisfied, then the geodesic equation is the same as (1), the Lorentz equation.

The mutual consistency of conditions such as (12) and (13) has been discussed extensively in Fontaine and Amiot (1983). The simplest approach is to exhibit a form

$$B_\mu = A_\mu + h\theta_{,\mu}$$

where $\theta(x^\mu)$ is a function that satisfies $v^\mu \theta_{,\mu} = 1$. The relation $H_{\mu\nu} = F_{\mu\nu}$ still holds due to the chain rule, $h_{,\nu} = (dh/d\theta)\theta_{,\nu}$. As pointed out in Fontaine and Amiot (1983), θ can be defined as a simple path-dependent integral of the proper time along the particle trajectory. Then the variable h is given by

$$h = -(e/mck) - A_\alpha v^\alpha \tag{14}$$

To summarise this section: The metric of (4) gives a prescription for measuring distances that produces a geodesic identical with the Lorentz equation.

Note that there has thus far been no restriction on the constant k. The identification of k will follow from the consideration of the field equations. It should be mentioned here that the same results as above are obtained if it is initially assumed that k is some function of the variables of the system.

Also, essentially the same results as above could have been obtained using a general gravitational metric $g_{\mu\nu}$ instead of $\eta_{\mu\nu}$. The metric $\eta_{\mu\nu}$ is taken not only for simplicity, but because it is reasonable to ignore purely gravitational effects in the realm of this study.

It is important to recognize now that the space being dealt with has a Finslerian form. This follows directly from the velocity dependence of the metric.

Finsler spaces have been known for a long time. The basic reference is still the book by Rund (1959). There has been only minimal interest in the past in applying these spaces to physics.

Recently, though, the number of papers on Finsler geometry applied to physics has been increasing. The interested reader should see, for example, the work of Tavakol and Van den Bergh (1986) and Aringazin and Asanov (1985) and references therein.

The metric being considered here, while velocity-dependent, actually reduces to a Riemann space due to (13). This can easily be seen by looking at the Finsler metric function

$$F(x^\mu, v^\mu) = (\bar{g}_{\mu\nu} v^\mu v^\nu)^{1/2} = (b^{-2}\eta_{\mu\nu} v^\mu v^\nu)^{1/2}$$

The Finsler metric is

$$f_{\alpha\beta} = \tfrac{1}{2}\partial^2 F^2/\partial v^\alpha \, \partial v^\beta$$

But since b is taken to be a constant, then $f_{\alpha\beta} = b^{-2}\eta_{\alpha\beta}$, which is the condition for the Finsler space to be Riemannian. So all of the present work is carried out in terms of familiar Riemannian geometry.

2. THE FIELD EQUATIONS

The Ricci tensor for the metric $\bar{g}_{\mu\nu}$ will be computed from (9) according to standard methods. The result is

$$
\begin{aligned}
R_{\eta\gamma} = & -\tfrac{1}{4}k^2\bar{g}^{\alpha\gamma}\bar{g}^{\tau\mu}F_{\tau\lambda}F_{\alpha\mu}B_\eta B_\gamma - \tfrac{1}{2}k\bar{g}^{\mu\lambda}F_{\gamma\lambda}F_{\eta\mu} \\
& -\tfrac{1}{2}k^2(1+kB^2)^{-1}\eta^{\mu\lambda}F_{\alpha\mu}B^\alpha(B_\eta B_{\lambda,\gamma}+B_\gamma B_{\lambda,\eta}) \\
& -\tfrac{1}{2}k^2(1+kB^2)^{-1}\bar{g}^{\mu\lambda}B_{\mu,\lambda}B^\alpha(B_\eta F_{\gamma\alpha}+B_\gamma F_{\eta\alpha}) \\
& +\tfrac{1}{2}k\bar{g}^{\mu\lambda}(F_{\eta\lambda,\mu}B_\gamma+F_{\gamma\lambda,\mu}B_\eta)-\tfrac{1}{2}k(1+kB^2)^{-1}\eta^{\mu\lambda}B_{\gamma,\lambda}B_{\eta,\mu} \\
& -k[\tfrac{1}{2}(1+kB^2)^{-1}\eta^{\mu\lambda}-k(1+kB^2)^{-2}B^\mu B^\lambda]B_{\lambda,\gamma}B_{\mu,\eta} \\
& +\tfrac{1}{2}k(1+kB^2)^{-1}\bar{g}^{\mu\lambda}B_{\mu,\lambda}(B_{\gamma,\eta}+B_{\eta,\gamma}) \\
& +\tfrac{1}{2}k(1+kB^2)^{-1}B^\alpha(F_{\alpha\gamma,\eta}+F_{\alpha\eta,\gamma})
\end{aligned}
\tag{15}
$$

The curvature scalar, $R=\bar{g}^{\eta\gamma}R_{\eta\gamma}$, is found to be

$$
\begin{aligned}
R = & -\tfrac{1}{4}k^2B^2(1+kB^2)^{-1}\bar{g}^{\alpha\lambda}\bar{g}^{\tau\mu}F_{\tau\lambda}F_{\alpha\mu}-\tfrac{1}{2}k\bar{g}^{\eta\gamma}\bar{g}^{\mu\lambda}F_{\gamma\lambda}F_{\eta\mu} \\
& +2k(1+kB^2)^{-1}\bar{g}^{\mu\lambda}B^\eta F_{\eta\lambda,\mu} \\
& -k(1+kB^2)^{-1}\bar{g}^{\mu\lambda}\bar{g}^{\eta\gamma}(B_{\gamma,\lambda}B_{\eta,\mu}-B_{\mu,\lambda}B_{\eta,\gamma}) \\
& -\tfrac{1}{2}k^2(1+kB^2)^{-2}\eta^{\mu\lambda}B^\eta B^\gamma F_{\eta\mu}F_{\gamma\lambda}
\end{aligned}
\tag{16}
$$

At this point notice that $R_{\eta\gamma}$ and R can be separated into terms of different order in the dimensionless magnitude kB^2. It will be shown a bit later that kB^2 is very large compared to unity, at least for a test particle with parameters comparable to those of an electron. So terms of different orders are grouped, making frequent use of $(1+kB^2)^{-1}\approx(kB^2)^{-1}-(kB^2)^{-2}$.

For example, to highest order,

$$
R=-\tfrac{1}{4}k\bar{g}^{\eta\gamma}\bar{g}^{\mu\lambda}F_{\gamma\lambda}F_{\eta\mu}
\tag{17}
$$

The Einstein tensor, $G_{\eta\gamma}=R_{\eta\gamma}-\tfrac{1}{2}\bar{g}_{\eta\gamma}R$, can be written as

$$
\begin{aligned}
G_{\eta\gamma} = & -\tfrac{1}{2}k^2\bar{g}^{\alpha\lambda}\bar{g}^{\tau\mu}F_{\tau\lambda}F_{\alpha\mu}B_\eta B_\gamma+\tfrac{1}{8}k\bar{g}_{\eta\gamma}\bar{g}^{\alpha\lambda}\bar{g}^{\tau\mu}F_{\alpha\mu}F_{\lambda\tau} \\
& -\tfrac{1}{2}k\bar{g}^{\mu\lambda}F_{\gamma\lambda}F_{\eta\mu}-\tfrac{1}{8}kB^{-2}\bar{g}^{\alpha\lambda}\bar{g}^{\tau\mu}F_{\alpha\mu}F_{\lambda\tau}B_\eta B_\gamma \\
& -\tfrac{1}{2}kB^{-2}\eta^{\mu\lambda}F_{\alpha\mu}B^\alpha(B_\eta B_{\lambda,\gamma}+B_\gamma B_{\lambda,\eta}) \\
& +\tfrac{1}{2}k\bar{g}^{\mu\lambda}(F_{\eta\lambda,\mu}B_\gamma+F_{\gamma\lambda,\mu}B_\eta)-kB^{-2}\bar{g}^{\mu\lambda}B^\alpha F_{\alpha\lambda,\mu}B_\eta B_\gamma \\
& +\tfrac{1}{2}kB^{-2}\bar{g}^{\mu\lambda}\bar{g}^{\alpha\tau}(B_{\alpha,\lambda}B_{\tau,\mu}-B_{\mu,\lambda}B_{\tau,\alpha})B_\eta B_\gamma \\
& +\tfrac{1}{4}kB^{-4}\eta^{\mu\lambda}B^\alpha B^\tau F_{\alpha\mu}F_{\tau\lambda}B_\eta B_\gamma
\end{aligned}
\tag{18}
$$

retaining terms of the two highest orders.

It is expected that the field equations for a particle in an electromagnetic field will be

$$
G_{\eta\gamma}=8\pi\kappa c^{-4}(\rho_0\bar{v}_\eta\bar{v}_\gamma+\bar{T}_{\eta\gamma})
\tag{19}
$$

with κ the gravitational constant and ρ_0 the proper matter density. The electromagnetic energy tensor is

$$\bar{T}_{\eta\gamma} = (4\pi)^{-1}(\bar{g}^{\alpha\lambda}F_{\eta\lambda}F_{\alpha\gamma} + \tfrac{1}{4}\bar{g}_{\eta\gamma}\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}) \tag{20}$$

The second and third terms of (18) compare exactly with $\bar{T}_{\eta\gamma}$ if

$$k = 4\kappa c^{-4} \tag{21}$$

So if (21) is accepted, then the electromagnetic energy tensor has appeared as part of the Einstein tensor. It is derived from the "geometry" as determined by the metric.

The $\bar{T}_{\eta\gamma}$ terms in (19) can be subtracted from both sides, leaving to highest order only

$$-\tfrac{1}{2}k^2\bar{g}^{\alpha\lambda}\bar{g}^{\tau\mu}F_{\tau\lambda}F_{\alpha\mu}B_\eta B_\gamma = 2\pi k\rho_0\bar{v}_\eta\bar{v}_\gamma$$

Recalling (17),

$$RB_\eta B_\gamma = \pi\rho_0\bar{v}_\eta\bar{v}_\gamma \tag{22}$$

Contracting this with $v^\eta v^\gamma$ gives

$$(e/mck)^2R = \pi\rho_0 b^{-2}c^4$$

or

$$(e/mc)^2R = 16\pi\kappa^2\rho_0 b^{-2}c^4 \tag{23}$$

An interesting point is that not only does the electromagnetic energy tensor part of the right-hand side of (19) appear in the curvature, but so does the matter term. Thus, one can say that everything in this theory is curvature.

Now that k has been determined, the magnitude assumption can be checked by looking at

$$b^{-2} = 1 + kc^{-2}(B_\alpha v^\alpha)^2 = 1 + \tfrac{1}{4}(e/m)^2\kappa^{-1} \approx 10^{42}$$

Substituting this in (22) and making use of (23) yields

$$kB_\eta B_\gamma = c^{-2}\bar{v}_\eta\bar{v}_\gamma \tag{24}$$

Since $\bar{v}^\mu = bv^\mu$ and $\bar{v}_\mu\bar{v}^\mu = c^2$, it can be inferred that the magnitude of $\bar{v}_\mu$ is of the order of $b^{-1}v_\mu$. Thus, $kB_\eta B_\gamma$ is of the same order of magnitude as b^{-2}. The number 10^{42} is about what might be expected when comparing electromagnetic and gravitational effects in the neighborhood of an electron.

So a consideration of the field equations yields several useful results, including an identification of the constant k, a "geometrization" of the energy-momentum tensor, and a relation between the matter energy density and the curvature scalar, which can be written as

$$R = 4\pi c^{-2}\kappa\rho_0 \tag{25}$$

This is not so startling when (17) is taken into account, producing

$$4\pi\rho_0 = c^{-2}\bar{g}^{\eta\gamma}\bar{g}^{\mu\lambda}F_{\gamma\lambda}F_{\eta\mu}$$

For example, if the field $F_{\mu\nu}$ is the self-field of the particle and the particle is taken to be in uniform motion,

$$4\pi\rho_0 = c^{-2}F^{\alpha\beta}F_{\alpha\beta}$$

This is a recognizable result. A recent appearance (in integral form) in the literature is in Schwinger (1983).

Finally, note that, due to (24), the metric can be written as

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + c^{-2}\bar{v}_\mu\bar{v}_\nu$$

This is the projection tensor form (Schmutzer, 1983), which also looks like the metric for a transparent medium (Synge, 1971).

3. COMPARISON WITH KALUZA-KLEIN FORM

The line element for the metric can be written

$$ds^2 = \bar{g}_{\mu\nu}\,dx^\mu\,dx^\nu = (\eta_{\mu\nu} + kA_\mu A_\nu)\,dx^\mu\,dx^\nu + 2khA_\mu\,dx^\mu\,d\theta + kh^2(d\theta)^2$$

This can be compared with the Kaluza-Klein form,

$$ds^2 = (\eta_{\mu\nu} + \beta^2 A_\mu A_\nu)\,dx^\mu\,dx^\nu - 2i\beta A_\mu\,dx^\mu\,dx^5 - (dx^5)^2$$

It is readily seen that a fundamental difference between the present theory and Kaluza theories is that the fifth dimension here would be timelike rather than spacelike.

The reason for the timelike fifth dimension is that part of the curvature derived from the metric is equated to the positive value of the electromagnetic energy tensor, which is kept on the right-hand side of the field equations. In Kaluza theories the electromagnetic part of the energy tensor is shifted from the right- to the left-hand side, where its negative value is equated to the curvature derived from the metric. The result here is a positive sign for k, which produces a 5-metric corresponding to a signature $(+1, -1, -1, -1, +1)$.

An objection that has been raised to the use of a timelike fifth dimension, and the reason for its avoidance in Kaluza-Klein theories [see Chyba (1985) and references therein], is that causality would be violated by the closed timelike lines in a compact dimension. The objection does not apply here, since the fifth dimension is not compact.

It should be pointed out that the five-dimensional interpretation given in this section is not essential to the theory. All results have been derived and presented in the curved 4-space and can be interpreted in ordinary general relativistic space-time.

4. DISCUSSION

The theory presented here establishes a correspondence between the electromagnetic field and the curvature of 4-space at a fundamental level. It is a unified theory in the same sense as Kaluza theories in that electromagnetism is incorporated directly into the metric structure of general relativity. It is not, however, totally unified since no common origin of the gravitational and electromagnetic fields has been given.

The present theory can be considered in a sense to be more unified than Kaluza theories. In Kaluza theories the matter density term remains alone on the right-hand side of the field equations and is not incorporated into the curvature derived from the metric. Here the matter density energy is included in the curvature. In a sense the field equations are reduced to a sort of tautology with the same physical interpretation for both sides of the equations—that is, everything is curvature.

Another interesting feature of this theory is the dependence of the metric on the motion of the test particle. This is not as radical as it first appears, since the test particle dependence is all in the gauge term. The usual physically meaningful quantities all involve only the gauge-independent field $F_{\mu\nu}$. There may, though, be a way of using ideas such as those of Apsel (1979) to give a measurable significance to the gauge.

The Finslerian nature of the space considered here should be explored further. Finsler space is a logical realm for the inclusion of electromagnetism in a metric theory, since it is intrinsically velocity-dependent. As pointed out by Tavakol and Van den Bergh (1986), it seems to be a natural geometric framework for an extension of general relativity. The properties of the present metric without the assumption (13) should be investigated.

Finally, the formal correspondence of this work to a theory with a timelike, noncompact fifth dimension indicates that it is fundamentally different from the usual theories of this type. It may be fruitful to consider this as a guide to quantum field theories in the same spirit as the recent intensive use of Kaluza-Klein theories.

ACKNOWLEDGMENTS

The author is indebted to the referee for suggesting the explicit mention of the Finslerian character of the metric.

REFERENCES

Apsel, David (1979). *General Relativity and Gravitation*, **10**, 297.
Aringazin, A. K., and Asanov, G. S. (1985). *General Relativity and Gravitation*, **17**, 1153.
Chyba, Christopher F. (1985). *American Journal of Physics*, **53**, 863.
Fontaine, Marie, and Amiot, Pierre (1983). *Annals of Physics*, **147**, 269.
Rund, H. (1959). *The Differential Geometry of Finsler Spaces*, Springer-Verlag, Berlin.
Schmutzer, Ernst (1983). In *Unified Field Theories of More than 4 Dimensions*, Venzo De Sabbata and Ernst Schmutzer, eds., World Scientific, Singapore, p. 81.
Schwinger, Julian (1983). *Foundations of Physics*, **13**, 373.
Synge, J. L. (1971). *Relativity, the General Theory*, North-Holland, Amsterdam, p. 383.
Tavakol, R. K., and Van den Bergh, N. (1986). *General Relativity and Gravitation*, **18**, 849.